18.100A Practice problems for Chapter 1-22,24,25

The final exam will take place on May 22nd, Tuesday 1:30-4:30.

As an open book exam, during the exam you can see
1. the textbook : *Introduction to Real Analysis* by A. Mattuck,
2. notes, copies, and scratch papers (at most 500 sheets of paper).

However, the following are NOT allowed to use
1. electronic devices.
2. the other books except the textbook.

When you write the poofs of problems, you can cite Theorems, Properties, and examples with proofs in the textbook Chapter 1-22,24,25. Moreover, a sheet of facts will be given and you can cite them.

However, you can not use exercises and problems in the textbook as well as problem sets, practice problems, and their solutions. If you have copies of the solutions and want to use them, please rewrite the proofs.

Review: 5 in pset 4, 2-10 in pset 5, 5-11 in pset 6, 1-5 in pset 7.

Problem 1. *Determine whether the following statements are true or false. If false then provide a counterexample. You don't need to verify why it is a counterexample.*

(1) *Suppose $f(x)$ is continuous on an interval I. Then, $f(x)$ is bounded on I.*
(2) *Suppose $\lim_{x\to 0^+} f(x) = +\infty$. Then, $f(x)$ is not continuous on $(0,1)$.*
(3) *Suppose $f(x)$ is continuous on an interval I. Then, $f(x)$ is uniformly continuous on I.*
(4) *Suppose $f(x)$ is continuous on $[0,1]$ and $f(0)<0$, $f(1)>0$. Then, $f(x)$ has a unique zero in $[0,1]$.*
(5) *Suppose $f(x)$ is continuous and bounded on $[0,+\infty)$. Then, $f(x)$ has the maximum on $[0,+\infty)$.*
(6) *Suppose $f(x)$ is infinitely many times differentiable at 0. Then, the Taylor series centered at 0 converges to $f(x)$ in a neighorhood of 0.*
(7) *Suppose $f(x)$ is a polynomial. Then, $f(x)$ is the same to its Taylor series.*
(8) *Suppose $\int_{0^+}^{1^-} f(x)dx$ converges. Then, $\int_{0^+}^{1^-} f^2(x)dx$ converges.*
(9) *Suppose $f(x)$ and $g(x)$ are integrable on $I=[a,b]$. Then, $f(x)g(x)$ is integrable on I.*
(10) *Suppose each $f_n(x)$ is bounded on $I=[a,b]$ and $f_n(x)$ converges to $f(x)$. Then, $f(x)$ is bounded on I.*
(11) *Let I_n are open intervals. Then, $\cap_{n=1}^{\infty} I_n$ is the empty set or an open interval.*
(12) *An open set U is not a union of closed sets.*

1. Taylor series

Problem 2. *Find the third order Taylor polynomial* $T_3(x)$ *of* $f(x) = e^x \sin x$ *at* 0, *and show that* $|f(x) - T_3(x)| < .02$ *for* $|x| < .5$.

(Fact : $\sqrt{e} < 1.75$.)

Problem 3. *Find the Taylor series of* $f(x) = x^3 + 2x^2 - 7$ *at* 1.

2. Continuity

Prove the following statements in this section.

Problem 4. $f(x, y)$ *is defined by* $f(x, y) = \sqrt{x^2 + y^2}\cos(\frac{1}{x^2+y^2})$ *for* $(x, y) \neq (0, 0)$ *and* $f(0, 0) = 0$. *Then,* $f(x, y)$ *is continuous at* $(0, 0)$.

Problem 5. $f(x, y)$ *is defined by* $f(x, y) = xy(x^2+y^2)^{-\frac{2}{3}}$ *for* $(x, y) \neq (0, 0)$ *and* $f(0, 0) = 0$. *Then,* $f(x, y)$ *is continuous at* $(0, 0)$.

Hint: $2xy \leq x^2 + y^2$.

Problem 6. $f(x, y)$ *is defined by* $f(x, y) = x \sin\big(\dfrac{1}{x^2 + y^2}\big)$ *for* $(x, y) \neq (0, 0)$ *and* $f(0, 0) = 0$. *Then,* $f(x, y)$ *is continuous at* $(0, 0)$.

Problem 7. $f(x, y)$ *is defined by* $f(x, y) = \sqrt{x^2 + y^2}\cos\big(1/y\big)$ *for* $y \neq 0$ *and* $f(x, 0) = 0$. *Then,* $f(x, y)$ *is not continuous.*

Problem 8. $f(x) = x\sin(1/x)$ *is uniformly continuous on* $(0, +\infty)$.

Problem 9. $f(x) = x^2 \sin(1/x)$ *is uniformly continuous on* $[1, +\infty)$.

Sorry. This problem is not well-designed and too difficult.

Problem 10. *Assume that* $g(x)$ *is continuous on* $\mathbb{R}$ *and* $g(0) = 1$. *Then,* $f(x) = g(x)\cos(1/x)$ *is not uniformly continuous on* $(0, 1]$.

Hint: Theorem 23. (This theorem was an assignment.)

3. Improper integral

Problem 11. *Test the improper integral for convergence or divergence.*

$$\int_0^\infty \frac{x}{\sqrt{1 + x^4}}dx.$$

Problem 12. *Test the improper integral for convergence or divergence.*

$$\int_e^\infty \frac{x}{(\ln x)^2\sqrt{1 + x^4}}dx.$$

Problem 13. *Assume that* $\int_a^b f^2(x)dx$ *and* $\int_a^b g^2(x)dx$ *converges. Prove that the improper integral converges.*

$$\int_a^b f(x)g(x)dx.$$

Problem 14. *Assume that* $\int_0^1 f^2(x)dx$ *converges. Prove that the improper integral converges.*

$$\int_0^1 f(x)x^{-\frac{1}{4}}dx.$$

Hint: Use the result of the previous problem.

4. Uniform convergence

Prove the following statements in this section.

Problem 15. $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2(n+1)}$ *is continuous on* $\mathbb{R}$.

Problem 16. $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2(n+1)}$ *is uniformly continuous on* $\mathbb{R}$.

Hint: First, show that $|f'(x)|$ is bounded.

Problem 17. $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2(x^2+1)}$ *attains its maximum on* $\mathbb{R}$.

Hint: First, show that $\lim_{|x|\to+\infty} f(x) = 0$.

5. Analysis in $\mathbb{R}^2$

Problem 18. *Assume that a continuous function* $f(x, y)$ *satisfies*

$$f(r, q) = r - q$$

for all $r, q \in \mathbb{Q}$. *Prove that* $f(x, y) = x - y$ *holds for all* $(x, y) \in \mathbb{R}^2$.

Problem 19. *Assume that a positive continuous function* $f(x, y)$ *satisfies* $\lim_{\|(x,y)\|\to+\infty} f(x, y) = 0$. *Prove that* $f(x, y)$ *attains its maximum, while it does not have its minimum.*

Problem 20. $f(x, y) = \frac{xy}{x^2 + y^2 + 1}$ *is uniformly continuous on* $\mathbb{R}^2$.

Note: It would be a bit challenging to prove the statement above.

Theorem 21. *Assume that $f(x, y)$ is uniformly continuous on $S \subset \mathbb{R}^2$, and $\{(a_n, b_n)\}$ is a Cauchy sequence in S. Then, $\{f(a_n, b_n)\}$ is a Cauchy sequence.*

Theorem 22. *Let $S = \{(x, y) : x^2 + 4y^2 \leq 9, x \geq 1\}$ and $f(x, y)$ is continuous on S. Show that f is bounded on S.*

6. Theorems

The following theorems are not given in the textbook, but you can cite during the final.

Theorem 23 (Pset 5, problem 5-(a))**.** *Assume $f(x)$ is uniformly continuous on I, and $\{a_n\}$ is a Cauchy sequence in I. Then, $\{f(a_n)\}$ is a Cauchy sequence.*

Theorem 24. $\sin x$ *and* $\cos x$ *are continuous on* $\mathbb{R}$*. Moreover, they has continuous derivatives* $(\sin x)' = \cos x$ *and* $(\cos x)' = -\sin x$*, respectively.*

Theorem 25. *Given two real numbers $x < y$, there exists a rational number r such that $x < r < y$. Moreover, given real number x, there exists a sequence $\{r_n\}$ of rational numbers such that* $\lim r_n = x$*.*